Two-Group Neutron Transport Theory with Anisotropic Scattering

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Two-group transport theory with anisotropic scattering in infinite homogeneous media is presented in this paper. The kernel of the integral equation is expanded into a finite series of Legendre polynomials. Eigenfunctions and eigenvalues of the transformed integral equation are found and the number of discrete eigenvalues is calculated. The full-range completeness theorem as well as the orthogonality and normalization relations are presented. As an example the expansion coefficients of the infinite-medium Green's function are explicitly calculated.

1. Introduction

The one-group neutron transport with anisotropic scattering in infinite homogeneous media has been considered by MIKA [1] by application of an expansion into Legendre polynomials of the scattering kernel and the transformed differential neutron flux. In an alternative approach, HEJTMANEK [2] has proposed, to write the kernel in the form of power series.

The two-group neutron transport with isotropic scattering in infinite homogeneous media has been dealt with by SIEWERT and SHIEH [3]. An extension to the multi-group neutron transport with isotropic scattering was given by YOSHIMURA and KATSURAGI [4].

In order to obtain a realistic treatment of neutron transport, we wish to synthesize both aspects by considering both, energy dependence and anisotropy of scattering. A first step in this direction has been taken by SIEWERT and FRALEY [5] for a special form of the scattering kernel.

In this paper, the theory of two-group neutron transport with anisotropic scattering in infinite homogeneous media will be presented, generalizing the formalism given by SIEWERT and SHIEH [2] through introduction of an expansion into Legendre polynomials of the scattering kernel and the transformed neutron flux.

In Section 2, eigenfunctions and eigenvalues of the two-group transport equation will be presented. The completeness of the set of eigenfunctions will be proved in Section 3. Orthogonality properties and normalisation integrals will be presented in Section 4. As an example expansion coefficients of the Green's function for a plane source in an infinite medium will be calculated in Section 5. In Appendix A, the number of discrete eigenvalues will be calculated. The agreement of the eigenvalue spectra of the two-group transport equation with those of its adjoint equation will be proved in Appendix B.

2. Eigenfunctions and Eigenvalues

Stationary neutron transport with anisotropic scattering in plane geometry is described by the Boltzmann equation

$$\mu \cdot \frac{\partial}{\partial x} \psi(x,\mu) + \Sigma\, \psi(x,\mu) = \sum_{k=0}^{N} B_k\, p_k(\mu) \int_{-1}^{+1} p_k(\mu')\; \psi(x,\mu')\, \mathrm{d}\mu' \tag{2.1}$$

after introduction of the spherical harmonics method. Application of the separation ansatz

$$\psi(x,\mu) = e^{-x/\eta} \cdot F(\eta,\mu) \tag{2.2}$$

delivers the transformed equation

$$(\eta\, \Sigma - \mu) \cdot F(\eta,\mu) = \eta \sum_{k=0}^{N} B_k\, p_k(\mu) \int_{-1}^{+1} p_k(\mu') \cdot F(\eta,\mu')\, \mathrm{d}\mu' . \tag{2.3}$$

With the help of the expansion coefficients

$$K_k(\eta) = \int_{-1}^{+1} p_k(\mu) \cdot F(\eta,\mu)\, \mathrm{d}\mu \tag{2.4}$$

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Eq. (3.7) may be written in the form

$$\varepsilon\,\psi'(\mu) = \varepsilon\,\mu \sum_{k=0}^{N} P_k(\mu)\, B_k\, H_k(\mu) \int_{-1}^{+1} \frac{1}{\eta-\mu}\, Y(\eta)\, \mathrm{d}\eta + D(\mu)\, Y(\mu), \tag{3.13}$$

where

$$\varepsilon\, D(\mu)\, Y(\mu) = D(\mu)\, Y(\mu) \tag{3.14}$$

is applied. Considering the boundary values $D^+(\mu)$ and $D^-(\mu)$ given in Appendix A, Eq. (3.13) is obtained in the form

$$2\,\varepsilon\,\psi'(\mu) = [D^+(\mu) - D^-(\mu)]\, \frac{1}{\pi i} \int_{-1}^{+1} \frac{1}{\eta-\mu}\, Y(\eta)\, \mathrm{d}\eta + [D^+(\mu) + D^-(\mu)] \cdot Y(\mu). \tag{3.15}$$

This equation will now be solved for $Y(\mu)$.

The function

$$N(z) = \frac{1}{\operatorname{Det} D(z)} \cdot D_c(z) \cdot \frac{1}{2\pi i} \int_{-1}^{+1} \frac{1}{\eta - z}\, \varepsilon(\eta)\, \psi'(\eta)\, \mathrm{d}\eta \tag{3.16}$$

is introduced, where $D_c(z)$ is the co-matrix of $D(z)$, i. e.,

$$D_c(z) = \begin{pmatrix} d_{22}(z) & -d_{12}(z) \\ -d_{21}(z) & d_{11}(z) \end{pmatrix}, \qquad D^{-1}(z) = \frac{1}{\operatorname{Det} D(z)} \cdot D_c(z)\,. \tag{3.17}$$

$\psi'(\eta)$ as given by Eq. (3.12), depends on the discrete coefficients a_i and the constants Γ_m. If the coefficients a_i are chosen such that

$$D_c(\eta_i) \int_{-1}^{+1} \frac{1}{\eta - \eta_i}\, \varepsilon(\eta)\, \psi'(\eta)\, \mathrm{d}\eta = 0 \qquad \text{for} \quad i = 1, 2, \ldots, 2M \tag{3.18}$$

[which is the condition of solubility for Eq. (3.15)], the function $N(z)$ is analytical in the whole plane cut along the real axis from -1 to $+1$. Applying Plemelj's formula to Eq. (3.16) leads to

$$\varepsilon \cdot \psi'(\mu) = D^+(\mu)\, N^+(\mu) - D^-(\mu)\, N^-(\mu)\,, \tag{3.19}$$

which may be written

$$2\,\varepsilon\,\psi'(\mu) = [D^+(\mu) - D^-(\mu)] \cdot [N^+(\mu) + N^-(\mu)] + [D^+(\mu) + D^-(\mu)] \cdot [N^+(\mu) - N^-(\mu)]\,. \tag{3.20}$$

Considering the analytical properties of $N(z)$ and its discontinuity along the cut from -1 to $+1$, one may write

$$N(z) = \frac{1}{2\pi i} \int_{-1}^{+1} \frac{1}{\eta - z}\, [N^+(\eta) - N^-(\eta)]\, \mathrm{d}\eta\,, \tag{3.21}$$

and therefore

$$N^+(\mu) + N^-(\mu) = \frac{1}{\pi i} \int_{-1}^{+1} \frac{1}{\eta - \mu}\, [N^+(\eta) - N^-(\eta)]\, \mathrm{d}\eta\,. \tag{3.22}$$

By insertion of (3.22) into (3.20) one finds

$$2\,\varepsilon\,\psi'(\mu) = [D^+(\mu) - D^-(\mu)] \cdot \frac{1}{\pi i} \int_{-1}^{+1} \frac{1}{\eta - \mu}\, [N^+(\eta) - N^-(\eta)]\, \mathrm{d}\eta + [D^+(\mu) + D^-(\mu)] \cdot [N^+(\mu) - N^-(\mu)]\,. \tag{3.23}$$

Obviously, the function $N^+(\eta) - N^-(\eta)$ is the solution of the integral Eq. (3.15). Therefore

$$Y(\eta) = N^+(\eta) - N^-(\eta)\,. \tag{3.24}$$

The constants Γ_m are finally found by application of $\int_{-1}^{+1} \mathrm{d}\eta\, \eta^m$ to (3.24), from which the system of linear equations

$$\int_{-1}^{+1} Y(\eta)\, \eta^m\, \mathrm{d}\eta = \Gamma_m = \int_{-1}^{+1} \eta^m [N^+(\eta) - N^-(\eta)]\, \mathrm{d}\eta \tag{3.25}$$

is obtained, which under the condition of solubility may be solved for Γ_m.

The uniqueness of the solution (3.24) may be demonstrated by calculating the function $N(z)$ from a given solution $Y(\eta)$ and showing that $N(z)$ has the form of (3.16).

4. Orthogonality and Normalization of Eigenfunctions

The adjoint system of integral equations is defined as follows

$$(\eta \Sigma - \mu) \cdot F^\dagger(\eta, \mu) = \eta \sum_{k=0}^{N} \tilde{B}_k p_k(\mu) \int p_k(\mu') F^\dagger(\eta, \mu') \, d\mu', \quad (4.1)$$

where $\tilde{B}_k$ is the transported of B_k. The spectra of eigenvalues of Eq. (2.3) and of Eq. (4.1) agree, as is shown in Appendix B. Eigenfunctions of the transposed Eq. (4.1) are obtained from those given in Section 2 after substitution of matrices B_k by the transposed matrices.

Equation (2.3), which may be written in the form

$$\Sigma \cdot F(\eta, \mu) - \sum_{k=0}^{N} p_k(\mu) B_k \int_{-1}^{+1} p_k(\mu') F(\eta, \mu') \, d\mu' = (\mu/\eta) F(\eta, \mu), \quad (4.2)$$

is multiplied from the left by $\tilde{F}^\dagger(\eta', \mu)$ and integrated over μ from -1 to $+1$. Similarly, the transposed Eq. (4.1)

$$F^\dagger(\eta', \mu) \cdot \Sigma - \sum_{k=0}^{N} \int_{-1}^{+1} p_k(\mu') \cdot \tilde{F}^\dagger(\eta', \mu') \, d\mu' \cdot p_k(\mu) B_k = (\mu/\eta) \tilde{F}^\dagger(\eta', \mu) \quad (4.3)$$

is multiplied from the right by $F(\eta, \mu)$ and integrated over μ from -1 to $+1$. Subtracting the resulting equations yields

$$0 = \left(\frac{1}{\eta} - \frac{1}{\eta'}\right) \int_{-1}^{+1} \tilde{F}^\dagger(\eta', \mu) \cdot \mu \cdot F(\eta, \mu) \, d\mu, \quad (4.4)$$

therefore

$$\int_{-1}^{+1} \tilde{F}^\dagger(\eta', \mu) \, \mu \, F(\eta, \mu) \, d\mu = 0 \quad \text{for } \eta' \neq \eta. \quad (4.5)$$

Since the eigenvalue spectra are identical, this theorem may be used for the determination of the expansion coefficients without solving the singular integral Eq. (3.1).

The normalization integrals,

$$N_i = \int_{-1}^{+1} \tilde{F}^\dagger(\eta_i, \mu) \, \mu \, F(\eta_i, \mu) \, d\mu, \quad i = 1, 2, \dots, 2M, \quad (4.6)$$

are found by integration

$$N_i = \sum_{k=0}^{N} \tilde{K}_{k-1}^\dagger(\eta_i) \, k \, K_k(\eta_i) + \frac{N+1}{2} \tilde{K}_N^\dagger(\eta_i) \, K_{N+1}(\eta_i) \quad (4.7)$$

$$+ \frac{(N+1)^2}{2} [\tilde{K}_N^\dagger(\eta_i) \, C_i K_N(\eta_i) - 2 \tilde{K}_N^\dagger(\eta_i) \, \eta_i C_i K_{N+1}(\eta_i) + \tilde{K}_{N+1}^\dagger(\eta_i) \, C_i K_{N+1}(\eta_i)] \quad \text{for} \quad N \geqq 1,$$

where

$$C_i = \eta_i \Sigma [(\eta_i \Sigma)^2 - 1]^{-1}. \quad (4.8)$$

The following relations are useful for further applications: If $A(\eta)$ is an arbitrary function satisfying the Hölder condition in the interval $[-1, +1]$, relations

$$\int_{-1}^{+1} \mu \tilde{F}^\dagger(\eta', \mu) \int_{①} A(\eta) F_k^{(1)}(\eta, \mu) \, d\eta \, d\mu = A(\eta') \, \eta' \tilde{K}_0^\dagger(\eta') [\tilde{D}^\dagger(\eta') D(\eta') + \tilde{W}^\dagger(\eta') W(\eta')] K_{0k}^{(1)}(\eta') \, \Theta_1(\eta') \quad (4.9)$$

and

$$\int_{-1}^{+1} \mu \tilde{F}^\dagger(\eta', \mu) \int_{②} A(\eta) F^{(2)}(\eta, \mu) \, d\eta \, d\mu = A(\eta') \, \eta' \tilde{K}_0^\dagger(\eta') [\tilde{D}^\dagger(\eta') \, \varepsilon(\eta') \, D(\eta') + \tilde{W}^\dagger(\eta') \, \varepsilon(\eta') \, W(\eta')] K_0^{(2)}(\eta') \, \Theta_2(\eta') \quad (4.10)$$

are obtained with help of the Poincaré-Bertrand formula.

5. Green's Function for Infinite Media

As an example for solutions of the two-group transport equation with anisotropic scattering in infinite homogeneous media, a Green's function will be given.

If the source term is given by

$$S(x_0, \mu) = \delta(x_0) \cdot \begin{pmatrix} \delta(\mu - \mu_1) \\ \delta(\mu - \mu_2) \end{pmatrix}, \quad (5.1)$$

the Green's function $G(x, \mu)$ satisfies the boundary condition

$$\mu \cdot [G(x_0+0, \mu) - G(x_0-0, \mu)] = \begin{pmatrix} \delta(\mu-\mu_1) \\ \delta(\mu-\mu_2) \end{pmatrix} = S^*(\mu) \tag{5.2}$$

and

$$\lim_{x \to \pm\infty} G(x, \mu) = 0 . \tag{5.3}$$

The expansion of the Green's function may be written

$$G(x, \mu) = \sum_{i=1}^{M} a_{i+} F(\eta_i, \mu)\, e^{-x/\eta_i} + \int_0^{1/\sigma} \alpha_1(\eta)\, e^{-x/\eta} F_1^{(1)}(\eta, \mu)\, d\eta$$
$$+ \int_0^{1/\sigma} \alpha_2(\eta)\, e^{-x/\eta} F_2^{(1)}(\eta, \mu)\, d\eta + \int_{1/\sigma}^{1} \beta(\eta)\, e^{-x/\eta} F^{(2)}(\eta, \mu)\, d\eta \quad \text{for} \quad x > x_0 \tag{5.4}$$

and

$$G(x, \mu) = -\sum_{i=1}^{M} a_{i-} F(-\eta_i, \mu)\, e^{+x/\eta_i} - \int_{-1/\sigma}^{0} \alpha_1(\eta)\, e^{-x/\eta} F_1^{(1)}(\eta, \mu)\, d\eta$$
$$- \int_{-1/\sigma}^{0} \alpha_2(\eta)\, e^{-x/\eta} F_2^{(1)}(\eta, \mu)\, d\eta - \int_{-1}^{-1/\sigma} \beta(\eta)\, e^{-x/\eta} F^{(2)}(\eta, \mu)\, d\eta \quad \text{for} \quad x < x_0 ,$$

where (5.3) has been taken into account. Through application of (5.2), the singular integral equation

$$\frac{1}{\mu} S^*(\mu) = \sum_{i=1}^{M} a_{i+} F(\eta_i, \mu)\, e^{-x_0/\eta_i} + \sum_{i=1}^{M} a_{i-} F(-\eta_i, \mu)\, e^{+x_0/\eta_i} + \int_{①} \alpha_1(\eta)\, e^{-x_0/\eta} F_1^{(1)}(\eta, \mu)\, d\eta$$
$$+ \int_{①} \alpha_2(\eta)\, e^{-x_0/\eta} F_2^{(1)}(\eta, \mu)\, d\eta + \int_{②} \beta(\eta)\, e^{-x_0/\eta} F^{(2)}(\eta, \mu)\, d\eta \quad \text{for} \quad -1 \leqq \mu \leqq 1 \tag{5.5}$$

is obtained, the solutions of which are found with help of the orthogonality relations considered above.

The discrete coefficients are found by multiplication of Eq. (5.5) by $\widetilde{F}^{\dagger}(\pm\eta_k, \mu)$ from the left and integration over μ from -1 to $+1$

$$a_{k\pm} = e^{\pm x_0/\eta_k} \cdot \frac{1}{N_{k\pm}} \cdot \int_{-1}^{+1} \widetilde{F}^{\dagger}(\pm\eta_k, \mu)\, S^*(\mu)\, d\mu . \tag{5.6}$$

Choosing

$$K_{01}^{(1)} = K_{01}^{\dagger(1)} = \binom{1}{0} , \quad K_{02}^{(1)} = K_{02}^{\dagger(1)} = \binom{0}{1} \tag{5.7}$$

one finds by an analogous procedure

$$\begin{pmatrix} \alpha_1(\eta) \\ \alpha_2(\eta) \end{pmatrix} = \frac{1}{\eta} e^{x_0/\eta} \cdot [\widetilde{D}^{\dagger}(\eta)\, D(\eta) + \widetilde{W}^{\dagger}(\eta)\, W(\eta)]^{-1} \cdot \int_{-1}^{+1} \begin{pmatrix} \widetilde{F}_1^{\dagger(1)}(\eta, \mu) \cdot S^*(\mu) \\ \widetilde{F}_2^{\dagger(1)}(\eta, \mu) \cdot S^*(\mu) \end{pmatrix} d\mu \tag{5.8}$$

and

$$\beta(\eta) = \frac{1}{\eta} e^{x_0/\eta} \cdot [(d_{11}^{\dagger}(\eta)\, w_{22}^{\dagger}(\eta) - d_{12}^{\dagger}(\eta)\, w_{21}^{\dagger}(\eta)) \cdot (d_{11}(\eta)\, w_{22}(\eta) - d_{12}(\eta)\, w_{21}(\eta))$$
$$+ (\operatorname{Det} D(\eta))^2]^{-1} \cdot \int_{-1}^{+1} \widetilde{F}^{\dagger(2)}(\eta, \mu) \cdot S^*(\mu)\, d\mu , \tag{5.9}$$

where

$$W(\eta) \equiv (w_{ij}(\eta)) = \pi\, \eta \sum_{k=0}^{N} P_k(\eta)\, B_k\, H_k(\eta) . \tag{5.10}$$

Appendix A

Introducing the complex function

$$\Omega(z) = \operatorname{Det}\, [H_0 - 2z \sum_{k=0}^{N} Q_k(z)\, B_k\, H_k(z)] \tag{A.1}$$

of a complex variable z, the discrete eigenvalues η_i of the transformed Eq. (2.3) are determined by the zeros of $\Omega(z)$, i. e.,

$$\Omega(\eta_i) = 0 . \tag{A.2}$$

The number of the zeros of $\Omega(z)$ can be written as $2M$, since $\Omega(z)$ is an even function of z, as is easily proved by considering the parity properties of the matrices $Q_k(z)$ and $H_k(z)$:

$$Q_k(-z) = (-1)^{k+1} \cdot Q_k(z) , \tag{A.3}$$

$$H_k(-z) = (-1)^k \cdot H_k(z) . \qquad \text{(A.4)}$$

Furthermore, from the definition of $Q_k(z)$ and $H_k(z)$, $\Omega(z)$ is an analytic function in the whole plane cut along the real axis from -1 to $+1$. Therefore, the number of its zeros can be expressed by

$$2\,M = \frac{1}{2\,\pi}\,\Delta_0 \arg \Omega(z), \qquad \text{(A.5)}$$

where $\Delta_0 \arg \Omega(z)$ denotes the change of the argument of $\Omega(z)$ along a way encircling the cut.

Application of Plemelj's formula to (A.1) yields

$$\lim_{z\to\eta} \Omega(z) \equiv \Omega^{\pm}(\eta) \qquad \text{(A.6)}$$
$$= \mathrm{Det}[D(\eta) \ \pm i\,\varepsilon(\eta)\,W(\eta)] \quad \text{for} \quad -1 \leqq \eta \leqq 1,$$

where according to (5.10)

$$W(\eta) = \pi\,\eta \sum_{k=0}^{N} P_k(\eta)\;B_k\,H_k(\eta). \qquad \text{(A.7)}$$

With the help of Eq. (A.6) one may write Eq. (A.5)

$$2\,M = \frac{1}{2\,\pi}\,[\Delta_{-1,+1} \arg \Omega^{+}(\eta) + \Delta_{+1,-1} \arg \Omega^{-}(\eta)]\,, \qquad \text{(A.8)}$$

where $\Delta_{-1,\,+1} \arg \Omega(\eta)$ is the change of the argument of $\Omega^{+}(\eta)$ from -1 to $+1$ and $\Delta_{+1,\,-1} \arg \Omega^{-}(\eta)$ is the change of the argument of $\Omega^{-}(\eta)$ from $+1$ to -1. Since

$$\Omega^{+}(\eta) = \Omega^{-}(-\eta), \qquad \text{(A.9)}$$
$$\arg \Omega^{+}(\eta) = -\arg \Omega^{+}(-\eta), \qquad \text{(A.10)}$$
$$\arg \Omega^{+}(0) = \ \arg \Omega^{-}(0) = 0, \qquad \text{(A.11)}$$

one finds

$$M = \frac{1}{\pi} \arg \Omega^{+}(1). \qquad \text{(A.12)}$$

Appendix B

From the definition of the adjoint integral Eq. (4.1) it is easily seen that the continuous eigenvalues are the same as those of the integral Eq. (2.3). The discrete eigenvalues are defined by the zeros of the determinants *

$$|\,H_0 - 2\,z \sum_{k=0}^{N} Q_k(z)\;B_k\,H_k(z)\,| = 0 \qquad \text{(B.1)}$$

and

$$|\,H_0^{\dagger} - 2\,z \sum_{k=0}^{N} Q_k(z)\;\widetilde{B}_k\,H_k^{\dagger}(z)\,| = 0 \qquad \text{(B.2)}$$

respectively, where the matrices $H_k^{\dagger}(z)$ satisfy the recurrence relation (2.9) with B_k replaced by $\widetilde{B}_k$, i. e.,

$$(k+1)\;H_{k+1}^{\dagger}(z) + z\,[2\,\widetilde{B}_k - (2\,k+1)\;\Sigma]\;H_k^{\dagger}(z) + k\,H_{k-1}^{\dagger}(z) = 0$$

and

$$H_{-1}^{\dagger}(z) \equiv 0, \quad H_0^{\dagger}(z) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv H_0(z). \qquad \text{(B.3)}$$

In order to show the agreement of the zeros of Eq. (B.1) and Eq. (B.2), the identity

$$|\,H_0 - 2\,z \sum_{k=0}^{N} Q_k(z)\;B_k\,H_k(z)\,| \equiv |\,H_0^{\dagger} - 2\,z \sum_{k=0}^{N} Q_k(z)\;\widetilde{B}_k\,H_k^{+}(z)\,| \qquad \text{(B.4)}$$

will be proved.

It will be useful to simplify the Eqs. (B.1) and (B.2). Using the recurrence relation for the Legendre functions of the second kind

$$(k+1)\;Q_{k+1}(z) \ - (2\,k+1)\;z\,\Sigma\,Q_k(z)' + k\,Q_{k-1}(z) = 0 \quad \text{for} \quad k = 1, 2, \ldots \qquad \text{(B.5)}$$

and the recurrence relation (2.9) for the matrices $H_k(z)$ one finds by induction from N to $N+1$

$$H_0 - 2\,z \sum_{k=0}^{N} Q_k(z)\;B_k\,H_k(z) \equiv -\,(N+1)\;[Q_{N+1}(z)\;H_N(z) \ - Q_N(z)\;H_{N+1}(z)] \qquad \text{(B.6)}$$

and analogously

$$H_0^{\dagger} - 2\,z \sum_{k=0}^{N} Q_k(z)\;\widetilde{B}_k\,H_k^{\dagger}(z) \equiv -\,(N+1)\;\;[Q_{N+1}(z)\;H_N^{\dagger}(z) - Q_N(z)\;H_{N+1}^{+}(z)]\,. \qquad \text{(B.7)}$$

Therefore, Eq. (B.4) may be written

$$|\,Q_{N+1}(z)\;H_N(z) \ - Q_N(z)\;H_{N+1}(z)\,| \equiv |\,Q_{N+1}(z)\;H_N^{\dagger}(z) - Q_N(z)\;H_{N+1}^{\dagger}(z)\,|\,. \qquad \text{(B.8)}$$

* For reasons of simplicity, the determinant of any matrix A is designated by $|\,A\,|$ in this appendix.

For any two matrices with two rows and columns, A and B,

$$|A+B| = |A| + |B| + \mathrm{Tr}[A_c \cdot B] \qquad \text{and} \qquad \mathrm{Tr}[A_c \cdot B] = \mathrm{Tr}[A \cdot B_c] \tag{B.9, 10}$$

holds, with the index c denoting the co-matrix and with the symbol Tr denoting the trace.

One may easily calculate

$$\begin{aligned} &|Q_{N+1} H_N - Q_N H_{N+1}| - |Q_{N+1} H_N^\dagger - Q_N H_{N+1}^\dagger| \\ &\quad = |Q_{N+1}| \cdot [|H_N| - |H_N^\dagger|] + |Q_N| \cdot [|H_{N+1}| - |H_{N+1}^\dagger|] - \mathrm{Tr}[M_N Q_{N+1,c} Q_N], \end{aligned} \tag{B.11}$$

where

$$M_k = H_{k+1} \cdot H_{k,c} - \widetilde{H}_{k,c}^\dagger \cdot \widetilde{H}_{k+1}^\dagger \quad \text{for} \quad k = 0, 1, 2, \ldots . \tag{B.12}$$

The right side of Eq. (B.11) may be evaluated as follows: Solving Eq. (2.9) and the transposed Eq. (B.3) for $(k+1)\, H_{k+1}$ and $(k+1)\, \widetilde{H}_{k+1}^\dagger$ respectively, and subtracing the determinants of both expressions yields

$$(k+1)^2 [|H_{k+1}| - |H_{k+1}^\dagger|] = |A_k| \cdot [|H_k| - |H_k^\dagger|] + k^2 [|H_{k-1}| - |H_{k-1}^\dagger|] + k\, \mathrm{Tr}[A_k \cdot M_{k-1}], \tag{B.13}$$

where

$$A_k = 2\, B_k - (2\, k+1)\, \Sigma . \tag{B.14}$$

A recurrence relation for the matrices M_k is found by multiplication of (2.9) from the right by $H_{k,c}$ and the transposed of (B.3) from the left by $\widetilde{H}_{k,c}^\dagger$ and subtraction:

$$(k+1)\, M_k = -[|H_k| - |H_k^\dagger|]\, A_k - k\, M_{k-1,c} \quad \text{for} \quad k = 0, 1, 2, \ldots . \tag{B.15}$$

Using (B.13) and (B.15) successively, one finds

$$|H_k| - |H_k^\dagger| \equiv 0 \qquad \text{and} \qquad M_k \equiv 0 \tag{B.16, 17}$$

for $k = 0, 1, 2, \ldots$. Thus, the right side of Eq. (B.11) vanishes identically.

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